Derived Category Automorphisms from Mirror Symmetry

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Abstract

Inspired by the homological mirror symmetry conjecture of Kontsevich [30], we construct new classes of automorphisms of the bounded derived category of coherent sheaves on a smooth quasi–projective variety. *MSC (2000): 18E30; 14J32.*

1 Introduction

Let X be a quasi–projective variety over an algebraically closed field $\bar{k}$, E a *proper* subvariety of X of codimension d (with i denoting the closed embedding $E \hookrightarrow X$), and $q : E \to Z$ a *flat* morphism. We assume that X, E and Z are *smooth* varieties of dimensions $n, n-d$ and $n-d-k$ respectively, as indicated in the diagram below.

$$\begin{tikzcd} E^{n-d} \arrow[r, hook, "i"'] \arrow[d, "q"'] & X^n \\ Z^{n-d-k} & \end{tikzcd} \tag{1.1}$$

We assume that there exists an invertible sheaf θ on Z, such that $q^*(\theta \overset{\mathbf{L}}{\otimes} \omega_Z^{-1}) \cong \mathbf{L}i^*\omega_X^{-1}$, where ω_Z and ω_X are the canonical sheaves of Z and X, respectively. Note that if ω_X is trivial, then $\theta = \omega_Z$ has the required property.

The goal of this note is to construct new classes of automorphisms of the bounded derived category of coherent sheaves on X, associated with the geometrical context sketched above. An automorphism of a triangulated category is an exact functor which induces a self–equivalence. The automorphisms defined here are determined by the EZ–spherical objects $\mathcal{E}$ (definition 2.1) in the bounded derived category of coherent sheaves of E that generalize the spherical objects introduced by Kontsevich [31], and Seidel and Thomas [40]. The main example and inspiration for this work is the case when X is a projective Calabi–Yau variety.

Following the work of Mukai [35] on actions of functors on derived categories of coherent sheaves, Bondal and Orlov [8], Orlov [36] and Bridgeland [9] established criteria that characterize the equivalences of derived categories of coherent sheaves (see remark 2.12). Their work showed that in many instances the group of derived self–equivalences (automorphisms) captures essential properties of the algebraic variety itself. Quite tellingly, the case of Calabi–Yau varieties turned out to be one of the most interesting and difficult to study. Similar techniques have emerged as useful tools in various other contexts, for

example, in the study of the McKay correspondence [12], birational geometry [10], heterotic string theory compactifications [3] and D–branes in string theory [19], [2].

The foundation for our line of investigation is provided by the homological mirror symmetry conjecture of Kontsevich [30] (see [32] for recent progress in this direction). The conjecture states that mirror symmetry should be viewed an as equivalence between "Fukaya A_∞ category" of a Calabi–Yau variety Y and the bounded derived category of coherent sheaves $\mathrm{D}(X)$ of the mirror Calabi–Yau variety X. The point of view of the present work stems from some ramifications of the general conjecture as presented by Kontsevich in [31]. Important results in this direction were obtained in the paper of Seidel and Thomas [40], in which the derived category braided automorphisms represented by Fourier–Mukai functors associated with spherical objects were analyzed and compared with the mirror generalized Dehn twists of Seidel [39]. An investigation (adapted to the toric geometry context) of the correspondence between the expected mirror automorphisms was pursued by the present author in [26].

In fact, the classes of automorphisms introduced in the present work were obtained with the significant guidance provided by the interpretation of the mirror symmetric calculations presented in [26]. As a general principle, the automorphisms of the bounded derived category of coherent sheaves on a Calabi–Yau variety X should be be mirrored by some automorphisms of the Fukaya category of the mirror Calabi–Yau variety Y. Some of the latter automorphisms are determined by loops in the moduli space of complex structures on Y and they will depend on the type of components of the discriminant locus in the moduli space of complex structures on Y that are surrounded by the loop. Our proposal states that for each component of the discriminant locus in the moduli space of complex structures on Y there is *a whole class* of automorphisms of the bounded derived category of coherent sheaves on X induced by the EZ–spherical objects associated with a diagram of the type (1.1). For example, in the toric case, the so–called "A–discriminantal hypersurface" of [21] (also called the "principal component" in [16], [26] or the "conifold locus" in physics)[1] in the moduli space of complex structures on Y corresponds to the class of spherical objects on X (in the sense of Seidel and Thomas) which is obtained in our context for $Z = \mathrm{Spec}(\bar{k})$ (example 3.1). At least in the Calabi–Yau case, the more general EZ–spherical objects introduced in the present work and their associated automorphisms are geometric manifestations of the so–called "phase transitions" in string theory [47], [4]. Elementary contractions in the sense of Mori theory are examples of these (see example 3.5) and, through mirror symmetry, they will correspond to the various other components of the discriminant locus in the moduli space of complex structures on Y. A precise statement about how this correspondence should work in the toric case was given in [26]. An analysis of the correspondence in the closely related language of D–branes in string theory has been recently presented by P. Aspinwall [2].

There are further questions that can be posed in this context. The structure of the group of automorphisms is quite intricate and in general difficult to handle. Nevertheless, it seems that the structure of the discriminant locus in moduli space of complex structures on Y encodes a lot of information about the group of automorphisms of the bounded derived category of coherent sheaves on X. As explained to me by Professors V. Lunts and Y. Manin (see the end of [34]), the motivic version of the group should be intimately related to the Lie algebra actions studied by Looijenga and Lunts [33]. The relationship has been established in the case of abelian varieties in [23]. A related proposal has been recently analyzed by B. Szendrői [41].

Another facet of the story concerns the mirror symmetric Fukaya category automorphisms. Are there any "EZ–generalized Dehn twists" associated to (special) Lagrangian vanishing cycles in Y that are not necessarily topological spheres S^n?

[1]Following a suggestion of Paul Aspinwall [2], we choose to call it the *primary* component of the discriminant locus.

The automorphisms that are introduced in this work are local in the sense that they twist only the "part" of the derived category $\mathrm{D}(X)$ that consists of objects supported on the subvariety E. The property of $\mathcal{E} \in \mathrm{D}(E)$ to be EZ–spherical for a configuration of the type shown in the diagram (1.1) does not depend on the ambient space X (with $\omega_X = 0$), but rather on the normal bundle $N_{E/X}$ (remark 3.6). The local character of the picture has direct links with the so–called "local mirror symmetry" frequently invoked by physicists [27], [14]. By using the theory of t–structures on triangulated categories [6], this rough idea of "locality" can be made more precise. There is now a related and very interesting proposal in physics made by Douglas [19] that inspired a remarkable construction in mathematics by Bridgeland [11] about how to express the stability of D–branes in string theory using the triangulated structure of derived categories.

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Addendum. This paper is (hopefully) an improved version of an earlier preprint. I am very grateful to the careful referees who found a gap in the earlier version that led to a major revision. Since the writing of the preprint, the derived category techniques have proved to be very useful in many situations in algebraic geometry and string theory. We make no attempt to relate this work to these developments, although there seem to be many interesting connections. We only mention the papers [42], [43], [17], [5], which are intimately related to the results of the present work and provide more applications than those presented in the last section of this paper.

1.1 Notation and other general considerations

A variety is an integral separated scheme of finite type over an algebraically closed field $\bar{k}$. All the schemes considered in this work are quasi–projective and Gorenstein (section V.9 in [24]), i.e. the structure sheaf is a dualizing sheaf. For a Gorenstein scheme W over $\bar{k}$, $\Gamma : W \to \mathrm{Spec}(\bar{k})$, of pure dimension w, the dualizing sheaf ω_W is invertible and has the property $\Gamma^!(k) = \omega_W[w]$. Proposition V.9.6 in [24] shows that, if E and Z are Gorenstein schemes of pure dimensions $n-d$ and $n-d-k$, respectively, and $q : E \to Z$ is flat, then the scheme $E \times_Z E$ is Gorenstein of pure dimension $n-d+k$.

For a quasi–projective scheme W, we denote by $\mathrm{D}(W)$ the bounded derived category of coherent sheaves on W (see [46], [18], [24], [22], [15], for the general theory of derived and triangulated categories and duality theory).

We use the commonly accepted notation for the derived functors between derived categories, and the notation $[n]$ for the shift by n functor in a triangulated category. The Verdier (derived) dual of an object $\mathcal{F} \in \mathrm{D}(W)$ will be denoted by $\mathcal{D}_W\mathcal{F}$, with $\mathcal{D}_W\mathcal{F} \in \mathrm{D}(W)$ defined by

$$\mathcal{D}_W\mathcal{F} := \mathbf{R}\mathcal{H}om_W(\mathcal{F}, \mathcal{O}_W).$$

For a proper morphism of noetherian schemes of finite Krull dimension $f : W \to V$, the construction of a right adjoint $f^!$ for $\mathbf{R}f_* : \mathrm{D}(W) \to \mathrm{D}(V)$ was originally established in the algebraic context by Grothendieck (see [24], [18], [45], [15]). We will use extensively the properties associated with the triple of adjoint functors $(\mathbf{L}f^*, \mathbf{R}f_*, f^!)$. Given $\mathcal{F} \in \mathrm{D}(W)$, $\mathcal{G} \in \mathrm{D}(V)$, we will also use the natural functorial

isomorphisms

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_W(\mathbf{L}f^*\mathcal{G},\mathcal{F}) \cong \mathbf{R}\mathcal{H}om_V(\mathcal{G},\mathbf{R}f_*\mathcal{F}) \qquad \text{([24] II.5.10)} \tag{1.2}$$

and

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_W(\mathcal{F},f^!\mathcal{G}) \cong \mathbf{R}\mathcal{H}om_V(\mathbf{R}f_*\mathcal{F},\mathcal{G}), \tag{1.3}$$

with the latter one provided by the Grothendieck duality theorem ([24] III.11.1).

In the concrete situation presented in the beginning of the introduction (diagram (1.1)), and under the given hypotheses, the functors $i^!$ and $q^!$ are given by

$$\begin{aligned} i^!(-) &\cong \mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes} i^!(\mathcal{O}_X) \cong \mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes} (\omega_E \overset{\mathbf{L}}{\otimes} \mathbf{L}i^*\omega_X^{-1})[-d], \\ q^!(-) &\cong q^*(-) \overset{\mathbf{L}}{\otimes} q^!(\mathcal{O}_Z) = q^*(-) \overset{\mathbf{L}}{\otimes} \omega_{E/Z}[k] \cong q^*(-) \overset{\mathbf{L}}{\otimes} (\omega_E \overset{\mathbf{L}}{\otimes} q^*\omega_Z^{-1})[k]. \end{aligned}$$

2 Main Results

We work under the assumptions presented in the beginning of the introduction. Our particular geometric context leads us to consider the following commutative diagram ($Y := E \times_Z E$). One could say that, in some sense, this note is about understanding the geometry summarized by this diagram.

$$\begin{tikzcd}
& & X \times X \arrow[dddll, bend right, "p_2"'] \arrow[dddrr, bend left, "p_1"] & & \\
& & E \times E \arrow[u, hook, "l"] \arrow[ddl, "r_2"'] \arrow[ddr, "r_1"] & & \\
& & Y \arrow[u, hook, "k"] \arrow[uu, hook, bend right, "j"'] \arrow[dl, "q_2"'] \arrow[dr, "q_1"] \arrow[dd, "t"] & & \\
X & E \arrow[l, hook', "i"] \arrow[dr, "q"] & & E \arrow[r, hook, "i"'] \arrow[dl, "q"'] & X \\
& & Z & &
\end{tikzcd} \tag{2.1}$$

We assume that there exists an invertible sheaf θ on Z such that

$$q^!\theta[-d-k] \cong i^!\mathcal{O}_X \cong \omega_{E/X}[-d], \tag{2.2}$$

which is the same thing as

$$q^*(\theta \overset{\mathbf{L}}{\otimes} \omega_Z^{-1}) \cong \mathbf{L}i^*\omega_X^{-1}. \tag{2.3}$$

In other words, the condition says that the pull–back of ω_X to E is also the pull-back of an invertible sheaf on Z. If $\mathbf{R}q_*(\mathcal{O}_E) \cong \mathcal{O}_Z$, and θ has the above property, the projection formula implies that $\theta \cong \omega_Z \overset{\mathbf{L}}{\otimes} \mathbf{R}q_*(\mathbf{L}i^*\omega_X^{-1})$.

2.1 EZ–spherical objects

Since $i : E \hookrightarrow X$ is a regular embedding of codimension d, $d \geq 0$, the normal sheaf of E in X, $\nu = (\mathcal{J}_E/\mathcal{J}_E^2)^\vee$, is locally free, and $\Lambda^d \nu[-d] \cong \omega_{E/X}[-d] \cong i^! \mathcal{O}_X$.

Definition 2.1. *An object $\mathcal{E} \in \mathrm{D}(E)$ is said to be EZ–***spherical** *if, either,*

i) $d = 0$ *and there exists a distinguished triangle*

$$\mathcal{O}_Z \longrightarrow \mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E}\big) \longrightarrow \theta[-k] \to \mathcal{O}_Z[1], \tag{2.4}$$

or,

ii) $d > 0$, *and*

$$\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E}\big) \cong \mathcal{O}_Z,\ \ \mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^c \nu\big) = 0,\ \ \textit{for } 0 < c < d. \tag{2.5}$$

Remark 2.2. Note that, in the case $d > 0$, the first condition in ii) implies that

$$\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^d \nu\big) \cong \theta[-k]. \tag{2.6}$$

Indeed, if

$$\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E}\big) \cong \mathcal{O}_Z,$$

then, Grothendieck duality implies that

$$\begin{aligned} &\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^d \nu\big) \cong \mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \omega_{E/X}\big) \cong \\ &\cong \mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}), q^! \theta[-k]\big) \cong \mathbf{R}\mathcal{H}om_Z\big(\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}), \theta[-k]\big) \cong \theta[-k]. \end{aligned} \tag{2.7}$$

Remark 2.3. Note also that in the case $d = 0$, an object $\mathcal{E} \in \mathrm{D}(X)$ such that

$$\mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, \mathcal{E}\big) \cong \mathcal{O}_Z \oplus \theta[-k]$$

is EZ–spherical. If $Z = \mathrm{Spec}(k)$, an EZ–spherical object is spherical in the sense of [40]. Note also the similarity with the notion of a simple morphism discussed there.

2.2 Kernels and Derived Correspondences

Given two schemes X_1, X_2, an object $\mathcal{G} \in \mathrm{D}(X_1 \times X_2)$, with finite Tor-dimension and proper support over both factors, determines an exact functor $\Phi_{\mathcal{G}} : \mathrm{D}(X_1) \to \mathrm{D}(X_2)$ by the formula ([35] prop. 1.4)

$$\Phi_{\mathcal{G}}(?) := \mathbf{R}p_{2_*}(\mathcal{G} \overset{\mathbf{L}}{\otimes} p_1^*(?)),$$

The object $\mathcal{G} \in \mathrm{D}(X_1 \times X_2)$ is called a *kernel*. By analogy with the classical theory of correspondences ([20] chap. 16), a kernel in $\mathrm{D}(X_1 \times X_2)$ is also said to be a *correspondence* from X_1 to X_2.

In the situation described by diagram (2.1), E is proper, so any object $\mathcal{G}' \in \mathrm{D}(E \times E)$ induces an exact functor $\Phi_{\mathcal{G}} : \mathrm{D}(X) \to \mathrm{D}(X)$ by

$$\Phi_{\mathcal{G}}(?) := \mathbf{R}p_{2_*}(\mathcal{G} \overset{\mathbf{L}}{\otimes} p_1^*(?)),$$

where $\mathcal{G} := l_*(\mathcal{G}') \in \mathrm{D}(X \times X)$.

By the projection formula (II.5.6 in [24]), we have that

$$i_*\Phi_{\mathcal{G}'}\mathbf{L}i^*(?) \cong i_*\mathbf{R}r_{2_*}(\mathcal{G}' \overset{\mathbf{L}}{\otimes} r_1^*\mathbf{L}i^*(?)) \cong \mathbf{R}p_{2_*}(\mathcal{G} \overset{\mathbf{L}}{\otimes} p_1^*(?)) \cong \Phi_{\mathcal{G}}(?).$$

We recall some well–known facts about kernels (correspondences) between bounded derived categories of coherent sheaves and their induced exact/Fourier–Mukai functors in the case of smooth quasi–projective varieties.

All the kernels that appear in this work have proper support over the two factors. The composition of the kernels (correspondences) $\mathcal{G}_{12} \in \mathrm{D}(X_1 \times X_2)$ and $\mathcal{G}_{23} \in \mathrm{D}(X_2 \times X_3)$ is defined by

$$\mathcal{G}_{23} \star \mathcal{G}_{12} := \mathbf{R}p_{13_*}\big(p_{12}^*(\mathcal{G}_{12}) \overset{\mathbf{L}}{\otimes} p_{23}^*(\mathcal{G}_{23})\big), \tag{2.8}$$

with $\mathcal{G}_{23} \star \mathcal{G}_{12} \in \mathrm{D}(X_1 \times X_3)$. In the case $X_1 = X_2 = X$, the neutral element for the composition of kernels in $\mathrm{D}(X \times X)$ is $\Delta_*\mathcal{O}_X$, where, as before, $\Delta : X \hookrightarrow X \times X$ is the diagonal morphism. It is well known ([35] prop. 1.3.) that there is a natural isomorphism of functors

$$\Phi_{\mathcal{G}_{23}\star\mathcal{G}_{12}}(?) \cong \Phi_{\mathcal{G}_{23}} \circ \Phi_{\mathcal{G}_{12}}(?).$$

From the formula (2.8), it follows that the functors $\Theta_{\mathcal{G}_{23}} : \mathrm{D}(X_1 \times X_2) \to \mathrm{D}(X_1 \times X_3)$ and $\Theta_{\mathcal{G}_{12}} : \mathrm{D}(X_2 \times X_3) \to \mathrm{D}(X_1 \times X_3))$ defined by

$$\Theta_{\mathcal{G}_{23}}(?) := \mathcal{G}_{23} \star (?), \Theta_{\mathcal{G}_{12}}(?) := (?) \star \mathcal{G}_{12},$$

are exact functor between triangulated categories and an argument similar to the proof of proposition 16.1.1. in [20] gives the associativity of the composition of kernels

$$\begin{aligned} &\mathcal{G}_{34} \star (\mathcal{G}_{23} \star \mathcal{G}_{12}) \cong (\mathcal{G}_{34} \star \mathcal{G}_{23}) \star \mathcal{G}_{12} \cong \\ &\cong \mathbf{R}(p_{14}^{1234})_*\Big((p_{12}^{1234})^*(\mathcal{G}_{12}) \overset{\mathbf{L}}{\otimes} (p_{23}^{1234})^*(\mathcal{G}_{23}) \overset{\mathbf{L}}{\otimes} (p_{34}^{1234})^*(\mathcal{G}_{34})\Big) \end{aligned} \tag{2.9}$$

with $\mathcal{G}_{12} \in \mathrm{D}(X_1 \times X_2)$, $\mathcal{G}_{23} \in \mathrm{D}(X_2 \times X_3)$, $\mathcal{G}_{34} \in \mathrm{D}(X_3 \times X_4)$.

We now return to the geometric context of the diagram (2.1). For any objects $\mathcal{E}, \mathcal{F} \in \mathrm{D}(E)$, we introduce the following objects in $\mathrm{D}(X \times X)$

$$\begin{aligned} \mathcal{R}(\mathcal{E}, \mathcal{F}) &:= j_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \ \overset{\mathbf{L}}{\otimes} \ q_2^*(\mathcal{F})\big), \quad \mathcal{R}\mathcal{E} := \mathcal{R}(\mathcal{E}, \mathcal{E}), \\ \mathcal{L}(\mathcal{E}, \mathcal{F}) &:= j_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} q^!\mathcal{O}_Z) \ \overset{\mathbf{L}}{\otimes} \ q_2^*(\mathcal{F})\big), \quad \mathcal{L}\mathcal{E} := \mathcal{L}(\mathcal{E}, \mathcal{E}), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &i^!\mathcal{O}_X = \omega_E \overset{\mathbf{L}}{\otimes} \mathbf{L}i^*\omega_X[-d], \\ &q^!\mathcal{O}_Z = \omega_{E/Z}[k] = \omega_E \overset{\mathbf{L}}{\otimes} q^*\omega_Z^{-1}[k]. \end{aligned}$$

Proposition 2.4. *For any $\mathcal{E}, \mathcal{F} \in \mathrm{D}(E)$, there exist functorial isomorphisms*

$$\begin{aligned} i) \qquad & \mathrm{Hom}_{X\times X}(\,?\star\mathcal{R}(\mathcal{E},\mathcal{F}),\ ??) \cong \mathrm{Hom}_{X\times X}(\,?,\ ??\star\mathcal{L}(\mathcal{F},\mathcal{E})), \\ & \mathrm{Hom}_{X\times X}(\mathcal{L}(\mathcal{E},\mathcal{F})\star ?,\ ??) \cong \mathrm{Hom}_{X\times X}(\,?,\mathcal{R}(\mathcal{F},\mathcal{E})\star ??). \end{aligned}$$

$$\begin{aligned} ii) \qquad & \mathrm{Hom}_{X\times X}(\,?\star\mathcal{L}(\mathcal{E},\mathcal{F}),\ ??) \cong \mathrm{Hom}_{X\times X}(\,?,\ ??\star\mathcal{R}(\mathcal{F},\mathcal{E})), \\ & \mathrm{Hom}_{X\times X}(\mathcal{R}(\mathcal{E},\mathcal{F})\star ?,\ ??) \cong \mathrm{Hom}_{X\times X}(\,?,\mathcal{L}(\mathcal{F},\mathcal{E})\star ??). \end{aligned}$$

Proof. This result is a version of a well known fact in the projective case (see for example Lemma 1.2 in [8]). In order to be able to use the adjunction theorems in our context, the lack of properness requires more care. We first prove that

$$\mathrm{Hom}_{X_1\times X_3}(\,?\star\mathcal{R}(\mathcal{E},\mathcal{F}),\ ??) \cong \mathrm{Hom}_{X_2\times X_3}(\,?,\ ??\star\mathcal{L}(\mathcal{F},\mathcal{E})),$$

where the labels for the different copies of X will help with the bookkeeping.

We first note that

$$?\star\mathcal{R}(\mathcal{E},\mathcal{F}) \cong \mathbf{R}p_{13_*}\big(p_{12}^*(j_{12_*}(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} p_{23}^*(?)\big),$$

with the appropriate object $\mathcal{R}'(\mathcal{E},\mathcal{F}) \in \mathrm{D}(Y_{12})$, $\mathcal{R}'(\mathcal{E},\mathcal{F}) = q_1^*(\mathcal{D}_{E_1}\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_{X_1}) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{F})$, $Y_{12} = E_1 \times_Z E_2$.

The fiber square

$$\begin{tikzcd} Y_{12}\times X_3 \arrow[r, "m_{12}"] \arrow[d, "(j_{12}, Id_{X_3})"'] & Y_{12} \arrow[d, "j_{12}"] \\ X_1\times X_2\times X_3 \arrow[r, "p_{12}"] & X_1\times X_2 \end{tikzcd} \tag{2.11}$$

shows that

$$p_{12}^*\ j_{12_*} \cong (j_{12}, Id_{X_3})_*\ m_{12}^*,$$

where m_{12} is the projection map $Y_{12}\times X_3 \to Y_{12}$. By the projection formula, we then obtain that

$$\begin{aligned} \mathbf{R}p_{13_*}\big(p_{12}^*(j_{12_*}(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} p_{23}^*(?)\big) &\cong \mathbf{R}p_{13_*}\ (j_{12}, Id_{X_3})_*\big(m_{12}^*(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} (j_{12}, Id_{X_3})^*\ p_{23}^*(?)\big) \cong \\ &\cong \mathbf{R}k_{13_*}\big(m_{12}^*(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} k_{23}^*(?)\big), \end{aligned} \tag{2.12}$$

where the *proper* maps $k_{13} : Y_{12}\times X_3 \to X_1\times X_3$ and $k_{23} : Y_{12}\times X_3 \to X_2\times X_3$ are defined as the compositions

$$k_{13} := p_{13}(j_{12}, Id_{X_3}), k_{23} := p_{23}(j_{12}, Id_{X_3}).$$

By adjunction, we obtain that

$$\begin{aligned} \mathrm{Hom}_{X_1\times X_3}\big(\,?\star\mathcal{R}(\mathcal{E},\mathcal{F}),\ ??\big) &\cong \mathrm{Hom}_{X_1\times X_3}\big(\mathbf{R}k_{13_*}\big(m_{12}^*(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} k_{23}^*(?)\big),\ ??\big) \cong \\ &\cong \mathrm{Hom}_{Y_{12}\times X_3}\big(m_{12}^*(\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} k_{23}^*(?), k_{13}^!(??)\big) \cong \\ &\cong \mathrm{Hom}_{Y_{12}\times X_3}\big(k_{23}^*(?), m_{12}^*(\mathcal{D}_{Y_{12}}\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} k_{13}^!(??)\big) \cong \\ &\cong \mathrm{Hom}_{X_2\times X_3}\big(\,?, \mathbf{R}k_{23_*}\big(m_{12}^*(\mathcal{D}_{Y_{12}}\mathcal{R}'(\mathcal{E},\mathcal{F})) \overset{\mathbf{L}}{\otimes} k_{13}^!(??)\big)\big). \end{aligned} \tag{2.13}$$

We have also used the fact that, since the projections $Y_{12} \to E_1$, $Y_{12} \to E_2$ are flat, and $\mathcal{E} \in \mathrm{D}(E_1)$, $\mathcal{F} \in \mathrm{D}(E_2)$, the object $\mathcal{R}'(\mathcal{E},\mathcal{F}) \in \mathrm{D}(Y_{12})$ is isomorphic to a bounded complex of locally free sheaves of finite rank (even though Y_{12} might be singular). The fiber square

$$\begin{tikzcd} Y_{12} \times X_3 \arrow[r, "m_{12}"] \arrow[d, "k_{13}"'] & Y_{12} \arrow[d, "u_1"] \\ X_1 \times X_3 \arrow[r, "p_1"] & X_1 \end{tikzcd} \tag{2.14}$$

with p_1 flat and u_1 proper, and the base change theorem (theorem 2 in [45], or theorem 5(i) in [29]) shows that

$$m_{12}^* u_1^! \mathcal{O}_{X_1} \cong k_{13}^! p_1^* \mathcal{O}_{X_1} \cong k_{13}^! \mathcal{O}_{X_1 \times X_3}.$$

By combining this fact with (2.13) and the adapted version of (2.12), we see that

$$\mathrm{Hom}_{X_1\times X_3}\big(\ ? \star \mathcal{R}(\mathcal{E},\mathcal{F}),\ ??\big) \cong \mathrm{Hom}_{X_2\times X_3}\big(?, \mathbf{R}p_{23_*}(p_{12}^*\big(j_{12_*}(\mathcal{D}_{Y_{12}}\mathcal{R}'(\mathcal{E},\mathcal{F}) \overset{\mathbf{L}}{\otimes} u_1^!\mathcal{O}_X)\big) \overset{\mathbf{L}}{\otimes} p_{13}^*(??))\big). \tag{2.15}$$

Recall that $\mathcal{R}'(\mathcal{E},\mathcal{F}) = q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \ \overset{\mathbf{L}}{\otimes}\ q_2^*(\mathcal{F})$. Hence, we can write that

$$\begin{aligned} \mathcal{D}_{Y_{12}}\mathcal{R}'(\mathcal{E},\mathcal{F}) \overset{\mathbf{L}}{\otimes} u_1^!\mathcal{O}_X &\cong \mathbf{R}\mathcal{H}om_{Y_{12}}(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \ \overset{\mathbf{L}}{\otimes}\ q_2^*(\mathcal{F}), \mathcal{O}_{Y_{12}}) \overset{\mathbf{L}}{\otimes} u_1^!\mathcal{O}_{X_1} \cong \\ &\cong q_1^*(\mathcal{E}) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{D}_E\mathcal{F} \overset{\mathbf{L}}{\otimes} q^!\mathcal{O}_Z), \end{aligned} \tag{2.16}$$

since

$$\mathcal{D}_{Y_{12}}\big(q_1^*(i^!\mathcal{O}_X)\big) \overset{\mathbf{L}}{\otimes} u_1^!\mathcal{O}_{X_1} \cong q_1^*(\omega_E^{-1} \overset{\mathbf{L}}{\otimes} \mathbf{L}i^*\omega_X[-d]) \overset{\mathbf{L}}{\otimes} (\omega_{Y_{12}} \overset{\mathbf{L}}{\otimes} q_1^*\mathbf{L}i^*\omega_X^{-1}[-d+k]) \cong q_2^*(q^!\mathcal{O}_Z), \tag{2.17}$$

where we use proposition II 5.8 in [24] (q_1 is flat) and the flat base change theorem quoted above. Combine this fact with (2.15), (2.16), to see that we have proved indeed that

$$\begin{aligned} \mathrm{Hom}_{X_1\times X_3}\big(\ ? \star \mathcal{R}(\mathcal{E},\mathcal{F}),\ ??\big) &\cong \mathrm{Hom}_{X_2\times X_3}\big(\ ?, \mathbf{R}p_{23_*}(p_{21}^*\mathcal{L}(\mathcal{F},\mathcal{E}) \overset{\mathbf{L}}{\otimes} p_{13}^*(??))\big) \cong \\ &\cong \mathrm{Hom}_{X_2\times X_3}\big(\ ?,\ ?? \star \mathcal{L}(\mathcal{F},\mathcal{E})\big). \end{aligned}$$

The proof of the second isomorphism of part i) is completely analogous.

The proof of the isomorphisms of part ii) is also very similar, but the existence of the invertible sheaf θ on Z enters the argument in a crucial way. For the first isomorphism of part ii), the only change occurs in formula (2.17) as follows:

$$\begin{aligned} \mathcal{D}_{Y_{12}}\big(q_1^*(q^!\mathcal{O}_Z)\big) \overset{\mathbf{L}}{\otimes} u_1^!\mathcal{O}_{X_1} &\cong q_2^*\omega_E[k] \overset{\mathbf{L}}{\otimes} q_1^*\mathbf{L}i^*\omega_X^{-1}[-d+k] \cong q_2^*\omega_E \overset{\mathbf{L}}{\otimes} t^*(\theta \overset{\mathbf{L}}{\otimes} \omega_Z^{-1})[-d] \cong \\ &\cong q_2^*\big(\omega_E \overset{\mathbf{L}}{\otimes} q^*(\theta \overset{\mathbf{L}}{\otimes} \omega_Z^{-1})[-d]\big) \cong q_2^*(i^!\mathcal{O}_X). \end{aligned}$$

The proof of the second isomorphism of part ii) is again analogous. □

The following easy result will be essential for our arguments.

Lemma 2.5. *Assume that $\mathcal{K} \in \mathcal{D}$ is an object in a triangulated category $\mathcal{D}$ such that there exists a collection of distinguished triangles (a "Postnikov system")*

$$\begin{tikzcd}
0=\mathcal{K}^{a-1} \arrow[rr] & & \mathcal{K}^{a} \arrow[dl] \arrow[rr] & & \mathcal{K}^{a+1} \arrow[dl] \arrow[r] & \cdots \arrow[r] & \mathcal{K}^{b-1} \arrow[rr] & & \mathcal{K}^{b}=\mathcal{K} \arrow[dl] \\
& \mathcal{H}^{a}(\mathcal{K})[-a] \arrow[ul, dotted] & & \mathcal{H}^{a+1}(\mathcal{K})[-(a+1)] \arrow[ul, dotted] & & & & \mathcal{H}^{b}(\mathcal{K})[-b] \arrow[ul, dotted] &
\end{tikzcd} \tag{2.18}$$

with $a \le b, a, b \in \mathbb{Z}$.

i) If $T : \mathcal{D} \to \mathcal{D}'$ is an exact functor (i.e. triangle preserving and commuting with the translations), with $\mathcal{D}'$ another triangulated category, such that

$$T(\mathcal{H}^c(\mathcal{K})) = 0, \text{ for } a < c < b,$$

then there exists a distinguished triangle

$$\begin{tikzcd}
T(\mathcal{H}^{a}(\mathcal{K})[-a]) \arrow[rr] & & T(\mathcal{K}) \arrow[dl] \\
& T(\mathcal{H}^{b}(\mathcal{K})[-b]) \arrow[ul, dotted] &
\end{tikzcd}$$

ii) If $H : \mathcal{D} \to \mathcal{A}$ is a cohomological functor (i.e. mapping triangles into long exact sequences), with $\mathcal{A}$ an Abelian category, such that

$$H(\mathcal{H}^c(\mathcal{K})[-c-p]) = 0, \text{ for } a < c \le b, p = 0, 1,$$

then

$$H(\mathcal{K}) \cong H(\mathcal{H}^a(\mathcal{K})[-a]).$$

The prototype of a Postnikov system associated to an object $\mathcal{K}$ of a triangulated category $\mathcal{D}$ endowed with a t-structure (definition 1.3.1 in [6]) appears as a succession of distinguished triangles of the form

$$\begin{tikzcd}
\tau_{<a}\mathcal{K} \arrow[rr] & & \tau_{\le a}\mathcal{K} \arrow[dl] \\
& \mathcal{H}^{a}(\mathcal{K})[-a] \arrow[ul, dotted] &
\end{tikzcd}$$

where $\tau_{<a}, \tau_{\le a}$ are the truncation functors, and $\mathcal{H}^a$ the cohomology functors with values in the heart of the t-structure. We will use this construction in what follows for the case of the bounded derived category of coherent sheaves endowed with the usual t-structure. Some other Postnikov systems to be used in this note are provided by the following lemma.

Lemma 2.6. *Let $i : T \hookrightarrow S$ be a regular embedding of codimension d, with T proper, and let ν be the normal sheaf of T in S.*

i) The exact functor $\mathbf{L}i^*i_* : \mathrm{D}(T) \to \mathrm{D}(T)$ is induced by the kernel

$$\Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T) \in \mathrm{D}(T \times T)$$

where $\Gamma_{12} : T \hookrightarrow T \times S$, and $\Gamma_{23} : T \hookrightarrow S \times T$, the graph (diagonal) embeddings determined by $i : T \hookrightarrow S$, and there exists a Postnikov system of the type (2.18) with $a = -d, b = 0$, and

$$\mathcal{K} = \Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T),\ \mathcal{H}^{-c}(\mathcal{K}) \cong \Delta_* \Lambda^c \nu^\vee,\ \textit{for } 0 \leq c \leq d,$$

where $\Delta : T \hookrightarrow T \times T$ is the diagonal embedding.

ii) For any object $\mathcal{G} \in \mathrm{D}(T)$ there exists a Postnikov system of the type (2.18) with $a = -d, b = 0$, and

$$\mathcal{K} = \mathbf{L}i^*i_*\mathcal{G},\ \mathcal{H}^{-c}(\mathcal{K}) \cong \mathcal{G} \overset{\mathbf{L}}{\otimes} \Lambda^c \nu^\vee,\ \textit{for } 0 \leq c \leq d.$$

Proof. The exact functor $\mathbf{L}i^*i_*$ is induced by the kernel

$$\Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T) \cong \mathbf{R}p_{13_*}\left(p_{12}^*\Gamma_{12_*}(\mathcal{O}_T) \overset{\mathbf{L}}{\otimes} p_{23}^*\Gamma_{23_*}(\mathcal{O}_T)\right).$$

Consider the fiber square (with p_{12} flat)

$$\begin{tikzcd} T \times T \arrow[r, "q_1"] \arrow[d, "(\Gamma_{12}, Id_T)"'] & T \arrow[d, "\Gamma_{12}"] \\ T \times S \times T \arrow[r, "p_{12}"] & T \times S \end{tikzcd} \tag{2.19}$$

Therefore $p_{12}^*\Gamma_{12_*}(\mathcal{O}_T) \cong (\Gamma_{12_*}, Id_{T_*})(\mathcal{O}_{T\times T})$. By the projection formula, we can then write

$$\Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T) \cong \mathbf{R}p_{13_*}(\Gamma_{12_*}, Id_{T_*})\Big((\mathbf{L}\Gamma_{12}^*, Id_T^*)\ p_{23}^*\Gamma_{23_*}(\mathcal{O}_T)\Big).$$

But $p_{13} \circ (\Gamma_{12}, Id_T) = Id_{T\times T}$, and $p_{23} \circ (\Gamma_{12}, Id_T) = (i, Id_T)$. Hence

$$\Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T) \cong (\mathbf{L}i^*, Id_T^*)\ \Gamma_{23_*}(\mathcal{O}_T).$$

Set $\iota := (i, Id_T) : T \times T \to S \times T$. Then $\Gamma_{23} = \iota \circ \Delta$, with $\Delta : T \hookrightarrow T \times T$ the diagonal embedding, and it follows that

$$\Gamma_{23_*}(\mathcal{O}_T) \star \Gamma_{12_*}(\mathcal{O}_T) \cong \mathbf{L}\iota^*\iota_*(\Delta_*\mathcal{O}_T).$$

Now note that there exists a Postnikov system in $\mathrm{D}(T \times T)$ of the type (2.18) with $\mathcal{K} = \mathbf{L}\iota^*\iota_*(\Delta_*\mathcal{O}_T)$, and $\mathcal{H}^{-c}(\mathcal{K}) \cong \mathbf{L}^c\iota^*\iota_*(\Delta_*\mathcal{O}_T)$. On $S\times T$ these sheaves are the Tor-sheaves $\mathcal{T}or_c^{\mathcal{O}_{S\times T}}(\iota_*(\Delta_*\mathcal{O}_T), \iota_*\mathcal{O}_{T\times T})$ and can be computed with the help of a local Koszul resolution of $\iota_*\mathcal{O}_T$ in $S \times T$ (see, for example, VII.2.5 in [7]). It follows that

$$\mathcal{H}^{-c}(\mathcal{K}) \cong \begin{cases} \Delta_*\Lambda^c\nu^\vee & \text{if } 0 \leq c \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

The lemma follows, since this Postnikov system on $T \times T$ is mapped by an exact functor into the required Postnikov system on T. $\square$

Remark 2.7. By combining the previous two lemmas, it follows that, for an EZ–spherical object $\mathcal{E} \in \mathrm{D}(E)$, there exists a distinguished triangle

$$\mathcal{O}_Z \longrightarrow \mathbf{R}q_* \mathbf{R}\mathcal{H}om_E\big(\mathcal{E}, i^! i_* \mathcal{E}\big) \longrightarrow \theta[-d-k] \longrightarrow \mathcal{O}_Z[1].$$

This remark will not be used in this paper.

Proposition 2.8. *If $\mathcal{E}$ in $\mathrm{D}(E)$ is EZ–spherical, then:*

i) For $e < d+k$,

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big) \cong \mathrm{Hom}_{X\times X}\big(\Delta_*\mathcal{O}_X[-e], \mathcal{L}\mathcal{E}\big) \cong \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z[e]\big).$$

In particular,

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big) \cong \mathrm{Hom}_{X\times X}\big(\Delta_*\mathcal{O}_X[-e], \mathcal{L}\mathcal{E}\big) \cong \begin{cases} \bar{k} & \textit{if } e = 0, \\ 0 & \textit{if } e < 0. \end{cases}$$

ii) For $e < 2(d+k)$,

$$\mathrm{Hom}_{X\times X}\big(\mathcal{L}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big) \cong \mathrm{Hom}_{X\times X}\big(\Delta_*\mathcal{O}_X[-e], \mathcal{R}\mathcal{E}\big) \cong \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\theta[e-d-k]\big).$$

In particular, for $e < d+k$,

$$\mathrm{Hom}_{X\times X}\big(\mathcal{L}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big) \cong \mathrm{Hom}_{X\times X}\big(\Delta_*\mathcal{O}_X[-e], \mathcal{R}\mathcal{E}\big) = 0.$$

iii) For $e < d+k$,

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \mathcal{R}\mathcal{E}[e]\big) \cong \mathrm{Hom}_{X\times X}\big(\mathcal{L}\mathcal{E}[-e], \mathcal{L}\mathcal{E}\big) \cong \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z[e]\big).$$

In particular,

$$\mathrm{Hom}_{X\times X}(\mathcal{R}\mathcal{E}, \mathcal{R}\mathcal{E}[e]) \cong \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E}[-e], \mathcal{L}\mathcal{E}) \cong \begin{cases} \bar{k} & \textit{if } e = 0, \\ 0 & \textit{if } e < 0. \end{cases}$$

Proof. Note that the first isomorphisms in each of the three parts of proposition follow directly from proposition 2.4, while the isomorphisms in the second groups of each part follow easily provided we proved the second isomorphisms in each part. The notation follows diagram (2.1).

i) By adjunction, we have that

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big) \cong \mathrm{Hom}_{E\times E}\Big(k_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E})\big), l^!\Delta_*\mathcal{O}_X[e]\Big). \tag{2.20}$$

Since $q_1 = r_1 \circ k$, $q_2 = r_2 \circ k$, the projection formula gives

$$k_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E})\big) \cong \big(r_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} r_2^*(\mathcal{E})\big) \overset{\mathbf{L}}{\otimes} k_*\mathcal{O}_Y. \tag{2.21}$$

On the other hand

$$l^!\Delta_*\mathcal{O}_X \cong \mathbf{L}l^*\Delta_*\mathcal{O}_X \overset{\mathbf{L}}{\otimes} l^!\mathcal{O}_{X\times X}. \tag{2.22}$$

Note that for $\mathbf{L}l^*\Delta_*\mathcal{O}_X$, there exists a Postnikov system in $\mathrm{D}(E\times E)$ of the type (2.18) with $\mathcal{K} = \mathbf{L}l^*\Delta_*\mathcal{O}_X$, and $\mathcal{H}^{-c}(\mathcal{K}) \cong \mathbf{L}^c l^*\Delta_*\mathcal{O}_X$. Indeed, on $X\times X$, these sheaves are in fact the Tor-sheaves $\mathcal{T}or_c^{\mathcal{O}_{X\times X}}(\Delta_*\mathcal{O}_X, l_*\mathcal{O}_{E\times E})$. Since $l_*\mathcal{O}_{E\times E} \cong p_1^*(i_*\mathcal{O}_E) \overset{\mathbf{L}}{\otimes} p_2^*(i_*\mathcal{O}_E)$ we have that

$$\mathbf{L}\Delta^* l_*\mathcal{O}_{E\times E} \cong i_*\mathcal{O}_E \overset{\mathbf{L}}{\otimes} i_*\mathcal{O}_E \cong i_*(\mathbf{L}i^* i_*\mathcal{O}_E),$$

so $\mathbf{L}^c\Delta^* l_*\mathcal{O}_{E\times E} \cong i_*\Lambda^c\nu^\vee$, for $0 \leq c \leq d$. On $X \times X$, these sheaves are again the Tor-sheaves $\mathcal{T}or_c^{\mathcal{O}_{X\times X}}(\Delta_*\mathcal{O}_X, l_*\mathcal{O}_{E\times E})$. Therefore, we see that

$$\mathcal{H}^{-c}(\mathcal{K}) \cong \begin{cases} \Delta_*\Lambda^c\nu^\vee & \text{if } 0 \leq c \leq d, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta : E \hookrightarrow E \times E$ is the diagonal embedding.

Our goal is to apply part $ii)$ of lemma 2.5 for the cohomological functor $H : \mathrm{D}(E) \to \mathbf{Ab}$ induced by the right hand side of (2.20),

$$H(\mathcal{K}) := \mathrm{Hom}_{E\times E}\Big(k_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E})\big), \mathcal{K} \overset{\mathbf{L}}{\otimes} l^!\mathcal{O}_{X\times X}[e]\Big).$$

where $\mathcal{K} = \mathbf{L}l^*\Delta_*\mathcal{O}_X$ with the Postnikov system described above, and $a = -d, b = 0$.

The hypotheses of part $ii)$ of lemma 2.5 require us to show that the group

$$\mathrm{Hom}_{E\times E}\Big(k_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E})\big), \Delta_*\Lambda^c\nu^\vee[c-p] \overset{\mathbf{L}}{\otimes} l^!\mathcal{O}_{X\times X}[e]\Big) \tag{2.23}$$

is zero for $0 \leq c < d, p = 0, 1$, and $e < d + k$.

To prove this claim, note first that the projection formula implies that

$$k_*\big(q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E})\big) \cong \big(r_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \overset{\mathbf{L}}{\otimes} r_2^*(\mathcal{E})\big) \overset{\mathbf{L}}{\otimes} k_*\mathcal{O}_Y,$$

and the already quoted flat base change theorem (theorem 2 in [45], or theorem 5(i) in [29]) shows that

$$l^!\mathcal{O}_{X\times X} \cong r_1^* i^!\mathcal{O}_X \overset{\mathbf{L}}{\otimes} r_2^* i^!\mathcal{O}_X. \tag{2.24}$$

We can then apply adjunction for the pair of functors $\big(\mathbf{L}\Delta^*, \Delta_*\big)$ associated to the diagonal of $E\times E$, to obtain that the group (2.23) is isomorphic to

$$\mathrm{Hom}_E\big(\mathbf{L}\Delta^*(k_*\mathcal{O}_Y) \overset{\mathbf{L}}{\otimes} (\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} \mathcal{E}), \Lambda^c\nu^\vee[c-p] \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X[e]\big) \cong$$
$$\cong \mathrm{Hom}_{E\times E}\big(k_*\mathcal{O}_Y, \Delta_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c}\nu\,[-d+c-p][e])\big).$$

Now consider the following commutative diagram

$$\begin{tikzcd} & Y \arrow[r, "t"] \arrow[d, "k"] & Z \arrow[d, "\Delta"] \\ E \arrow[ur, "\Delta_Y"] \arrow[r, "\Delta"] & E \times E \arrow[r, "q\times q"] & Z \times Z \end{tikzcd} \tag{2.25}$$

where $t = q_1 \circ q = q_2 \circ q$. Since $q \times q : E \times E \to Z \times Z$ is flat, the base change theorem for the above fiber square allows us to write

$$k_*\mathcal{O}_Y \cong k_*t^*\mathcal{O}_Z \cong (q\times q)^*\Delta_*(\mathcal{O}_Z). \tag{2.26}$$

By adjunction, we conclude that the group (2.23) is isomorphic to

$$\mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c}\nu\,[-d+c-p][e])\big). \tag{2.27}$$

Since $\mathcal{E}$ is EZ–spherical, we have that $\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c}\nu) \cong 0$ for $0 < c < d$. For $c = 0$, remark 2.2 shows that $\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d}\nu) \cong \theta[-k]$, the group above is zero if

$$-d - k - p + e < 0,$$

that is when $e < d + k$. This proves the above claim that the group (2.23) is zero for $0 \leq c < d, p = 0, 1$, and $e < d + k$.

Therefore, part ii) of lemma 2.5 implies indeed that $\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \Delta_*\mathcal{O}_X[e]\big)$ is isomorphic to the group obtained by setting $c = d, p = 0$ in (2.27), that is

$$\mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}[e])\big),$$

which is isomorphic (for an EZ–spherical object $\mathcal{E}$) to

$$\mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z[e]\big).$$

Note that in the case $d = 0$, the same conclusion is obtained without having to use lemma 2.5, since in this case the EZ-condition implies the existence of a distinguished triangle of the form

$$\begin{gathered}\ldots \to \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\theta[-k+e-1]\big) \to \mathrm{Hom}_{Z\times Z}(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z[e]) \to \\ \to \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}[e])\big) \cong \mathrm{Hom}_{X\times X}(\mathcal{R}\mathcal{E}, \Delta_*\mathcal{O}_X[e]) \to \\ \to \mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\theta[-k+e]\big) \to \ldots.\end{gathered}$$

ii) The proof of the second isomorphism in this part is almost identical to the previous argument; formula (2.24) has to be replaced by

$$l^!\mathcal{O}_{X\times X} \cong r_1^*q^!\mathcal{O}_Z \overset{\mathbf{L}}{\otimes} (r_1^*q^*\theta[-d-k] \overset{\mathbf{L}}{\otimes} r_2^*i^!\mathcal{O}_X),$$

since, by (2.2) $q^!\theta[-d-k] \cong i^!\mathcal{O}_X$.

iii) We now proceed with the proof of the second isomorphism in this part. We have that

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \mathcal{R}\mathcal{E}[e]\big) \cong \mathrm{Hom}_{E\times E}\big(\mathcal{Z}, l^!l_*\mathcal{Z}[e]\big) \cong \mathrm{Hom}_{E\times E}\big(\mathcal{Z}, \mathbf{L}l^*l_*\mathcal{Z}[e] \overset{\mathbf{L}}{\otimes} l^!\mathcal{O}_{X\times X}\big),$$

with $\mathcal{Z} \in \mathrm{D}(E\times E)$ defined by

$$\mathcal{Z} := \big(r_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X) \ \overset{\mathbf{L}}{\otimes} \ r_2^*(\mathcal{E})\big) \ \overset{\mathbf{L}}{\otimes} \ k_*\mathcal{O}_Y.$$

According to lemma 2.6, there exists a Postnikov system with $\mathcal{K} = \mathbf{L}l^*l_*\mathcal{Z}$, $a = -2d, b = d$ and

$$\mathcal{H}^{-c}(\mathcal{K}) \cong \mathcal{Z} \overset{\mathbf{L}}{\otimes} \Lambda^c\tilde{\nu}^\vee,$$

for $0 \leq c \leq 2d$. where $\tilde{\nu}$ is the normal sheaf of $E\times E$ in $X \times X$.

In order to use again part ii) of lemma 2.5 for the cohomological functor $H : \mathrm{D}(E\times E) \to \mathbf{Ab}$, induced by the right hand side of (2.2),

$$H(\mathcal{K}) := \mathrm{Hom}_{E\times E}(\mathcal{Z}, \mathcal{K} \overset{\mathbf{L}}{\otimes} l^!\mathcal{O}_{X\times X}[e]\Big).$$

with $\mathcal{K} = \mathbf{L}l^*l_*\mathcal{Z}$, $a = -2d, b = d$, we need to show that

$$\mathrm{Hom}_{E\times E}\big(\mathcal{Z}, \mathcal{Z}[e] \overset{\mathbf{L}}{\otimes} \Lambda^c \tilde{\nu}^\vee [c-p] \overset{\mathbf{L}}{\otimes} l^! \mathcal{O}_{X\times X}\big), \tag{2.28}$$

is zero for $0 \leq c < 2d, p = 0, 1$, and $e < d + k$. Moreover, the Künneth formula implies that it is enough to show that the groups

$$\mathrm{Hom}_{E\times E}\big(\mathcal{Z}, \mathcal{Z}[e] \overset{\mathbf{L}}{\otimes} (\Lambda^{c_1}\tilde{\nu_1}^\vee \otimes \Lambda^{c_2}\tilde{\nu_2}^\vee)[c_1 + c_2 - p] \overset{\mathbf{L}}{\otimes} l^! \mathcal{O}_{X\times X}\big), \tag{2.29}$$

are zero for $0 \leq c_1 + c_2 < 2d, p = 0, 1, e < d+k$. Here the normal sheaves $\tilde{\nu_1}$ and $\tilde{\nu_2}$ give the decomposition of the normal sheaf $\tilde{\nu}$ along the two directions corresponding to the embeddings $E_i \hookrightarrow X_i$.

After regrouping the various tensor products we obtain the the group (2.29) is isomorphic to

$$\mathrm{Hom}_{E\times E}\Big(k_*\mathcal{O}_Y, \big(r_1^*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_1}\nu) \overset{\mathbf{L}}{\otimes} r_2^*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_2}\nu\big) \overset{\mathbf{L}}{\otimes} k_*\mathcal{O}_Y[-2d + c_1 + c_2 - p][e]\Big).$$

By (2.26), we have that $k_*\mathcal{O}_Y \cong (q \times q)^*\Delta_*\mathcal{O}_Z$, so adjunction and the projection formula imply that the group above is isomorphic to

$$\mathrm{Hom}_{Z\times Z}\Big(\Delta_*\mathcal{O}_Z, \Delta_*\mathbf{R}t_*\big(q_1^*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_1}\nu) \overset{\mathbf{L}}{\otimes} q_2^*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_2}\nu)[-2d + c_1 + c_2 - p][e]\Big).$$

Since q is flat, the Künneth formula for the interior fiber square in the diagram (2.1) says that

$$\mathbf{R}t_*\big(q_1^*\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} q_2^*\mathcal{G}_2\big) \cong \mathbf{R}q_*\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} \mathbf{R}q_*\mathcal{G}_2,$$

for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{D}(E)$. Indeed,

$$\begin{aligned}\mathbf{R}q_*\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} \mathbf{R}q_*\mathcal{G}_2 &\cong \mathbf{R}q_*\big(\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} q^*\mathbf{R}q_*\mathcal{G}_2\big) \cong \mathbf{R}q_*\big(\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} \mathbf{R}q_{1_*}q_2^*\mathcal{G}_2\big) \\ &\cong \mathbf{R}(q \circ q_1)_*\big(q_1^*\mathcal{G}_1 \overset{\mathbf{L}}{\otimes} q_2^*\mathcal{G}_2\big).\end{aligned}$$

Hence, we have shown that the group (2.29) is isomorphic to

$$\mathrm{Hom}_{Z\times Z}\Big(\Delta_*\mathcal{O}_Z, \Delta_*\big(\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_1}\nu) \overset{\mathbf{L}}{\otimes} \mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c_2}\nu)\big)[-2d + c_1 + c_2 - p][e]\Big).$$

Clearly, if $\mathcal{E}$ is EZ–spherical, and $c_1 = c_2 = 0$, the group is

$$\mathrm{Hom}_{Z\times Z}\Big(\Delta_*\mathcal{O}_Z, \Delta_*(\theta \otimes \theta)[-2d - 2k - p][e]\Big),$$

which clearly is zero for $e < d + k$. We also have to consider the case $c_1 + c_2 = d$. Then the group is

$$\mathrm{Hom}_{Z\times Z}\Big(\Delta_*\mathcal{O}_Z, \Delta_*\theta[-d - k - p][e]\Big),$$

which is again zero for $e < d + k$. In all the other situations with $0 \leq c_1 + c_2 < 2d$, the group is zero. It is immediate then that part *ii*) of lemma 2.5 implies the required isomorphism

$$\mathrm{Hom}_{X\times X}\big(\mathcal{R}\mathcal{E}, \mathcal{R}\mathcal{E}[e]\big) \cong \mathrm{Hom}_{Z\times Z}\Big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z[e]\Big),$$

for $e < d + k$.

As in the proof of part *i*), the case $d = 0$ from the above calculation follows without invoking lemma 2.5. $\square$

For an EZ–spherical object $\mathcal{E} \in \mathrm{D}(E)$, set $e = 0$ in the isomorphisms of part $i)$ of the above proposition. Denote by $r_{\mathcal{E}}$ and $l_{\mathcal{E}}$ the images of the identity morphism under some choice of isomorphisms at part $i)$.

$$\mathcal{RE} \xrightarrow{r_{\mathcal{E}}} (\Delta_{X\times X})_*(\mathcal{O}_X), \quad (\Delta_{X\times X})_*(\mathcal{O}_X) \xrightarrow{l_{\mathcal{E}}} \mathcal{LE}.$$

Definition 2.9. *If $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical, the objects $\mathcal{E}_R$ and $\mathcal{E}_L$ in $\mathrm{D}(X \times X)$ are defined (up to non–canonical isomorphisms) by the distinguished triangles in $\mathrm{D}(X \times X)$*

$$\begin{aligned} &\mathcal{RE} \xrightarrow{r_{\mathcal{E}}} \Delta_*\mathcal{O}_X \to \mathcal{E}_R \to \mathcal{RE}[1] \\ &\mathcal{E}_L \to \Delta_*\mathcal{O}_X \xrightarrow{l_{\mathcal{E}}} \mathcal{LE} \to \mathcal{E}_L[1]. \end{aligned} \tag{2.30}$$

We will need the following corollary of proposition 2.8.

Proposition 2.10. *If $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical, then*

$$\mathrm{Hom}_{X\times X}(\mathcal{E}_L, \mathcal{LE}\,[-1]) \cong \mathrm{Hom}_{X\times X}(\mathcal{RE}, \mathcal{E}_R\,[-1]) = 0.$$

Proof. Apply the cohomological functor $\mathrm{Hom}_{X\times X}(?, \mathcal{LE})$ to the distinguished triangle

$$\mathcal{E}_L \to \Delta_*\mathcal{O}_X \xrightarrow{l_{\mathcal{E}}} \mathcal{LE} \to \mathcal{E}_L[1],$$

and look at the following piece of the resulting long exact sequence:

$$\begin{aligned} &\ldots \to \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X[1], \mathcal{LE}) \to \\ &\quad \to \mathrm{Hom}_{X\times X}(\mathcal{E}_L[1], \mathcal{LE}) \to \mathrm{Hom}_{X\times X}(\mathcal{LE}, \mathcal{LE}) \xrightarrow{l_{\mathcal{E}}^{\%}} \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \mathcal{LE}) \to \ldots. \end{aligned}$$

Note that, since $\mathcal{E}$ is EZ–spherical, by $i)$ and $iii)$ of proposition 2.8, the last two groups on the right are isomorphic to $\mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z\big)$ (and in fact to the field $\bar{k}$). The morphism $l_{\mathcal{E}}$ was chosen to correspond to the identity in the group $\mathrm{Hom}_{Z\times Z}\big(\Delta_*\mathcal{O}_Z, \Delta_*\mathcal{O}_Z\big)$, which shows that the induced homomorphism $l_{\mathcal{E}}^{\%}$ is in fact an isomorphism. Since part $i)$ of the previous proposition shows that the group $\mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X[1], \mathcal{LE})$ is zero, we obtain indeed that $\mathrm{Hom}_{X\times X}(\mathcal{E}_L[1], \mathcal{LE})$ is zero. The proof of the other half of the proposition is completely analogous. $\square$

The main result of this note is the following theorem.

Theorem 2.11. *Under the hypotheses of definition 2.1, for any EZ–spherical object $\mathcal{E} \in \mathrm{D}(E)$, the exact functors $\Phi_{\mathcal{E}_R}$ and $\Phi_{\mathcal{E}_L}$ are inverse automorphisms of $\mathrm{D}(X)$, i.e.*

$$\Phi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}_L} \cong Id_{\mathrm{D}(X)}, \qquad \Phi_{\mathcal{E}_L} \circ \Phi_{\mathcal{E}_R} \cong Id_{\mathrm{D}(X)}.$$

The proof of the theorem will occupy the next section.

Remark 2.12. At least in the projective case, a theorem of Bondal and Orlov (theorem 1.1 in [8]) and Bridgeland (theorem 5.1 [9]) provides a clear criterion for the fully faithfulness of the exact functors $\Phi_{\mathcal{E}_R}$ and $\Phi_{\mathcal{E}_L}$. Namely, $\Phi_{\mathcal{E}_R}$ is fully faithful if and only if, for each point $x \in X$,

$$\mathrm{Hom}_{X\times X}(\Phi_{\mathcal{E}_R}(\mathcal{O}_x), \Phi_{\mathcal{E}_R}(\mathcal{O}_x)) = \bar{k},$$

and for each pair of points $x_1, x_2 \in X$, and for each integer i,

$$\mathrm{Hom}_{X\times X}(\Phi_{\mathcal{E}_R}(\mathcal{O}_{x_1}), \Phi_{\mathcal{E}_R}(\mathcal{O}_{x_2})) = 0, \text{ unless } x_1 = x_2 \text{ and } 0 \le i \le n.$$

While this is a very geometrical characterization, the quite convoluted way of defining the objects $\mathcal{E}_R$ and $\mathcal{E}_L$ makes its use rather hard in the given set-up.

Remark 2.13. As it can be seen from the proofs contained in the next section, the smoothness assumptions on X, E and Z can be relaxed somewhat. Quasi–projectiveness can be replaced by assumptions on the schemes that guarantee that Grothendieck duality theory works. Beyond that, it is enough to assume that the schemes X, E, Z as well as the morphism q are *Gorenstein* ([24] V.9), with E proper, and that $E \hookrightarrow X$ is a regular embedding. We would then have to add "by hand" further assumptions on the kernels $\mathcal{E}$, $\mathcal{RE}$, $\mathcal{LE}$; for example, we could assume that they are *perfect,* namely that they are isomorphic (in the corresponding derived categories) to bounded complexes of locally free sheaves of finite rank. As it can be seen from the proof of proposition 2.8, we also need to assume that Z is connected. The results of this section would remain true. The crucial assumption that the morphism q is proper and flat ensures that the considered functors take *bounded* derived categories of *coherent* sheaves to *bounded* derived categories of *coherent* sheaves.

Remark 2.14. In the case of a smooth Calabi–Yau variety X ($\omega_X = 0, \theta = \omega_Z$), the choice $\mathcal{E} = \omega_E$ gives

$$\mathcal{RE} = j_*(\mathcal{O}_Y[-d] \overset{\mathbf{L}}{\otimes} q_2^*\omega_E).$$

The corresponding kernel $\mathcal{E}_R$ is precisely the kernel introduced in section 4.1 of [26] in the case of a Calabi–Yau complete intersection X in a toric variety (that result provided the inspiration for this work). As it is shown there, under the additional hypothesis that E is a complete intersection of toric divisors, the action in cohomology of the corresponding Fourier–Mukai functor $\Phi_{\mathcal{E}_R}$ matches the mirror symmetric monodromy action obtained by analytical computations. It will be seen in the section 3 of this work that, in that context, $\mathcal{E} = \omega_E$ is indeed EZ–spherical. The invertability of the corresponding exact functor in the case of a type III birational contraction for a Calabi–Yau threefold has been checked by B. Szendrői (see section 6.2 in [41]). In this special case, the cohomology action induced by the Fourier–Mukai transformation has been written down by P. Aspinwall (section 6.1 in [2]) based on string theoretic arguments (apparently the formula is a reinterpretation of previous physics results obtained in [28]). Miraculously (at least from the present author's point of view), it can be checked that Aspinwall's formula is in complete agreement with the results of this work.

2.3 Proof of the theorem

It is not surprising that in order to prove theorem 2.11 we need to compute the kernels $\mathcal{E}_R \star \mathcal{E}_L$, $\mathcal{E}_L \star \mathcal{E}_R$. As an intermediate step we will study the kernels $\mathcal{RE} \star \mathcal{LE}$ and $\mathcal{LE} \star \mathcal{RE}$. To that end, a few more facts about kernels (or correspondences) in $\mathrm{D}(Y)$, $Y = E \times_Z E$ are needed.

An object $\mathcal{G}_Y \in \mathrm{D}(Y)$ of finite Tor–dimension defines an exact functor from $\mathrm{D}(E)$ to $\mathrm{D}(E)$ by the formula

$$\Phi_{\mathcal{G}_Y}(\ ?) := \mathbf{R}q_{2_*}(\mathcal{G}_Y \overset{\mathbf{L}}{\otimes} q_1^*(\ ?)).$$

Note that we work under the important assumption that the morphism $q : E \to Z$ is flat. We can define the composition of two kernels $\mathcal{G}_Y$ and $\mathcal{F}_Y$ by the analog of the formula (2.8)

$$\mathcal{G}_Y \star \mathcal{F}_Y := \mathbf{R}q_{13_*}\big(q_{12}^*(\mathcal{F}_Y)\big) \overset{\mathbf{L}}{\otimes} q_{23}^*(\mathcal{G}_Y),$$

where the projection maps are flat morphisms from $E\times_Z E\times_Z E$ to $E\times_Z E$. All the properties mentioned above for the composition of kernels in $\mathrm{D}(X \times X)$ continue to hold in this case, most remarkably the associativity (due to the flatness of $q : E \to Z$).

Note also that we made the choice to denote the operation of composition of kernels in Y by the same symbol as before. It will be clear from the specific context which composition is meant in a particular formula.

The functors i_* and $\mathbf{L}i^*$ determined by the embedding $i : E \hookrightarrow X$ can be expressed as exact functors determined by the kernel $\Gamma_*(\mathcal{O}_E) \in \mathrm{D}(E \times X)$ viewed as a correspondence from E to X, or from X to E, respectively ($\Gamma : E \hookrightarrow E \times X$ is the graph embedding). It is a nice and easy exercise to check, using (2.9), that any kernel $l_*(\mathcal{F}) \in \mathrm{D}(X \times X)$ can be decomposed as

$$l_*(\mathcal{F}) \cong \Gamma_*(\mathcal{O}_E) \star \mathcal{F} \star \Gamma_*(\mathcal{O}_E), \tag{2.31}$$

($l : E \times E \hookrightarrow X \times X$ is the canonical embedding).

Lemma 2.15. *If $\mathcal{F}_Y, \mathcal{G}_Y \in \mathrm{D}(Y)$, then*

$$k_*(\mathcal{G}_Y) \star k_*(\mathcal{F}_Y) \cong k_*(\mathcal{G}_Y \star \mathcal{F}_Y),$$

where $k : Y \to E \times E$ is the canonical embedding.

Proof. (of the lemma)

We need to show that

$$k_*\mathbf{R}q_{13_*}\big(q_{12}^*(\mathcal{F}_Y) \overset{\mathbf{L}}{\otimes} q_{23}^*(\mathcal{G}_Y)\big) \cong \mathbf{R}r_{13_*}\big(r_{12}^*(k_*(\mathcal{F}_Y)) \overset{\mathbf{L}}{\otimes} r_{23}^*(k_*(\mathcal{G}_Y))\big).$$

Consider the fiber square (with r_{12} flat)

$$\begin{tikzcd} E_1 \times_Z E_2 \times E_3 \arrow[r, "m_{12}"] \arrow[d, "(k, Id_{E_3})"'] & E_1 \times_Z E_2 \arrow[d, "k"] \\ E_1 \times E_2 \times E_3 \arrow[r, "r_{12}"] & E_1 \times E_2 \end{tikzcd} \tag{2.32}$$

where we have used subscripts to distinguish between different copies of E. We have that $r_{12}^* k_* \cong (\mathbf{L}k^*, Id_{E_3}^*) m_{12}^*$. By the projection formula, we can write

$$\begin{aligned} &\mathbf{R}r_{13_*}\big(r_{12}^*(k_*(\mathcal{F}_Y)) \overset{\mathbf{L}}{\otimes} r_{23}^*(k_*(\mathcal{G}_Y))\big) \cong \\ &\cong \mathbf{R}r_{13_*}(k_*, Id_{E_3 *})\big(m_{12}^*(\mathcal{F}_Y) \overset{\mathbf{L}}{\otimes} (\mathbf{L}k^*, Id_{E_3}^*) r_{23}^*(k_*(\mathcal{G}_Y))\big) \end{aligned} \tag{2.33}$$

By using another fiber square (with $r_{23} \circ (k, Id_{E_3})$ flat)

$$\begin{tikzcd} E_1 \times_Z E_2 \times_Z E_3 \arrow[r, "q_{23}"] \arrow[d, "k_{12}"'] & E_2 \times_Z E_3 \arrow[d, "k"] \\ E_1 \times_Z E_2 \times E_3 \arrow[r, "r_{23} \circ (k, Id_{E_3})"] & E_2 \times E_3 \end{tikzcd} \tag{2.34}$$

Hence $(\mathbf{L}k^*, Id_{E_3}^*)r_{23}^* k_* \cong k_{12_*} q_{23}^*$, and again by the projection formula we obtain that the right hand side of (2.33) is isomorphic to

$$\begin{aligned}
&\mathbf{R}r_{13_*}\big(m_{12}^*(\mathcal{F}_Y) \overset{\mathbf{L}}{\otimes} k_*, Id_{E_3*})\big(k_{12_*} q_{23}^*(\mathcal{G}_Y)\big) \cong \\
&\cong \mathbf{R}r_{13_*}(k_{12}^* m_{12}^*(\mathcal{F}_Y) \overset{\mathbf{L}}{\otimes} k_*, Id_{E_3*})k_{12_*}\big(q_{23}^*(\mathcal{G}_Y)\big) \cong k_* \mathbf{R}q_{13_*}\big(q_{12}^*(\mathcal{F}_Y) \overset{\mathbf{L}}{\otimes} q_{23}^*(\mathcal{G}_Y)\big),
\end{aligned}$$

since $r_{13} \circ (k, Id_{E_3}) \circ k_{12} = k \circ q_{13}$. This ends the proof of the lemma. □

Proposition 2.16. *If $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical, there exist distinguished triangles*

$$\mathcal{LE} \longrightarrow \mathcal{RE} \star \mathcal{LE} \longrightarrow \mathcal{RE} \longrightarrow \mathcal{LE}\,[1], \tag{2.35}$$
$$\mathcal{LE} \longrightarrow \mathcal{LE} \star \mathcal{RE} \longrightarrow \mathcal{RE} \longrightarrow \mathcal{LE}\,[1]. \tag{2.36}$$

Proof. Write

$$\mathcal{RE} = j_* \mathcal{RE}_Y, \mathcal{LE} = j_* \mathcal{LE}_Y,$$

with

$$\mathcal{RE}_Y := q_1^*(\mathcal{D}_E \mathcal{E} \overset{\mathbf{L}}{\otimes} i^! \mathcal{O}_X) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E}),\ \mathcal{LE}_Y := q_1^*(\mathcal{D}_E \mathcal{E} \overset{\mathbf{L}}{\otimes} q^! \mathcal{O}_Z) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E}).$$

By (2.31), lemma 2.15 and the associativity of the composition of correspondences we can write that

$$\begin{aligned}
\mathcal{RE} \star \mathcal{LE} &= j_*(\mathcal{RE}_Y) \star j_*(\mathcal{LE}_Y) \cong \\
&\cong \Gamma_*(\mathcal{O}_E) \star k_*(\mathcal{RE}_Y) \star \big(\Gamma_*(\mathcal{O}_E) \star \Gamma_*(\mathcal{O}_E)\big) \star k_*(\mathcal{LE}_Y) \star \Gamma_*(\mathcal{O}_E)
\end{aligned}$$

We can now interpret the last line of the previous formula as an exact functor $T : \mathrm{D}(E \times E) \to \mathrm{D}(X \times X)$ given by

$$T(\mathcal{K}) := \Gamma_*(\mathcal{O}_E) \star k_*(\mathcal{RE}_Y) \star \mathcal{K} \star k_*(\mathcal{LE}_Y) \star \Gamma_*(\mathcal{O}_E).$$

Lemma 2.6 shows that there exists a Postnikov system of the type (2.18) with $a = -d, b = 0, \mathcal{K} = \Gamma_*(\mathcal{O}_E) \star \Gamma_*(\mathcal{O}_E)$, and $\mathcal{H}^{-c}(\mathcal{K}) \cong \Delta_* \Lambda^c \nu^\vee$, for $0 \leq c \leq d$, where $\Delta : E \hookrightarrow E \times E$ is the diagonal embedding.

In order to apply part $i)$ of lemma 2.5, we need to examine the action of the functor T on the cohomology sheaves $\mathcal{H}^{-c}$. We have that

$$\begin{aligned}
T(\mathcal{H}^{-c}[c]) &= \Gamma_*(\mathcal{O}_E) \star k_*(\mathcal{RE}_Y) \star \Delta_* \Lambda^c \nu^\vee[c] \star k_*(\mathcal{LE}_Y) \star \Gamma_*(\mathcal{O}_E) \cong \\
&\cong \Gamma_* \mathcal{O}_E \star k_*\big(\mathcal{RE}_Y \star \Delta_{Y*} \Lambda^c \nu^\vee[c] \star \mathcal{LE}_Y\big) \star \Gamma_* \mathcal{O}_E,
\end{aligned}$$

where we have used lemma 2.15, and the fact that $\Delta = k \circ \Delta_Y$ (see diagram (2.25)).

We have that

$$(\mathcal{RE}_Y \star \Delta_{Y*} \Lambda^c \nu^\vee[c]) \star \mathcal{LE}_Y \cong \mathbf{R}q_{13_*}\Big(q_{12}^* \mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} q_{23}^* \mathcal{RE}_Y \star \Delta_{Y*} \Lambda^c \nu^\vee[c]\Big).$$

But

$$q_{23}^* \mathcal{RE}_Y \star \Delta_{Y*} \Lambda^c \nu^\vee[c] \cong (q_2^{123})^*(\mathcal{D}_E \mathcal{E} \overset{\mathbf{L}}{\otimes} (\Lambda^c \nu^\vee[c] \overset{\mathbf{L}}{\otimes} i^! \mathcal{O}_X)) \overset{\mathbf{L}}{\otimes} (q_3^{123})^*(\mathcal{E}) \cong$$

$$q_{12}^* \mathcal{LE}_Y \cong (q_1^{123})^*(\mathcal{D}_E \mathcal{E} \overset{\mathbf{L}}{\otimes} q^! \mathcal{O}_Z) \overset{\mathbf{L}}{\otimes} (q_2^{123})^*(\mathcal{E}).$$

Since

$$\Lambda^c\nu^\vee[c] \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X \cong \Lambda^{d-c}\nu[-d+c], \tag{2.37}$$

after regrouping the terms, we obtain that

$$\begin{aligned}
&\mathcal{RE}_Y \star \Delta_{Y*}\Lambda^c\nu^\vee[c] \star \mathcal{LE}_Y \cong \mathbf{R}q_{13_*}\big(q_{13}^*\mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} (q_3^{123})^*(\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c}\nu[-d+c])\big) \cong \\
&\cong \mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} \mathbf{R}q_{13_*}(q_3^{123})^*(\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^d\nu[-d+c]) \cong \\
&\cong \mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} t^*\big(\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes} \Lambda^{d-c}\nu[-d+c])\big).
\end{aligned}$$

Since $\mathcal{E}$ is EZ–spherical, the last line in the above formula is zero unless $c = 0, d$. For $c = d$, the last line is $\mathcal{LE}_Y$, and for $c = 0$, it is, by remark 2.2,

$$\mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} t^*\theta[-d-k] \cong q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} q^!\mathcal{O}_Z \overset{\mathbf{L}}{\otimes} q^*\theta[-d-k]) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E}) \cong \mathcal{RE}_Y, \tag{2.38}$$

since $q^!\mathcal{O}_Z \overset{\mathbf{L}}{\otimes} q^*\theta[-d-k] \cong i^!\mathcal{O}_X$, by the starting assumption (2.2) made on θ. Part $i)$ of lemma 2.5 implies then that there exist a distinguished triangle

$$T(\mathcal{H}^{-d}[d]) \cong \mathcal{LE} \longrightarrow T(\Gamma_*\mathcal{O}_E \star \Gamma_*\mathcal{O}_E) = \mathcal{RE} \star \mathcal{LE} \longrightarrow T(\mathcal{H}^0) \cong \mathcal{RE} \longrightarrow \mathcal{LE}[1].$$

Note that the proof works also in the case $d = 0$. In that case, lemma 2.5 is not needed, and the above calculation shows that

$$\mathcal{RE} \star \mathcal{LE} \cong T(\mathcal{O}_X) \cong \mathcal{LE}_Y \overset{\mathbf{L}}{\otimes} t^*\big(\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}\big),$$

and the distinguished triangle (2.4) defining the EZ–condition in this case finishes the argument.

The proof of the existence of the second distinguished triangle is very similar, and requires the reversals of the roles of $\mathcal{RE}$ and $\mathcal{LE}$, the replacement of the observation (2.37) by

$$\Lambda^c\nu^\vee[c] \overset{\mathbf{L}}{\otimes} q^!\mathcal{O}_Z \cong \Lambda^{d-c}\nu[-d+c] \overset{\mathbf{L}}{\otimes} q^*\theta^{-1}[d+k],$$

and the use of the required analog of (2.38), namely

$$\mathcal{RE}_Y \overset{\mathbf{L}}{\otimes} t^*\theta^{-1}[d+k] \cong q_1^*(\mathcal{D}_E\mathcal{E} \overset{\mathbf{L}}{\otimes} i^!\mathcal{O}_X \overset{\mathbf{L}}{\otimes} q^*\theta^{-1}[d+k]) \overset{\mathbf{L}}{\otimes} q_2^*(\mathcal{E}) \cong \mathcal{LE}_Y.$$

□

Proposition 2.17. *If $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical, then*

$$\mathcal{E}_L \star \mathcal{RE}\ [1] \cong \mathcal{LE} \cong \mathcal{RE} \star \mathcal{E}_L\ [1].$$

Proof. For the first isomorphism, note that the operation $\star$ with one argument fixed is an exact functor of triangulated categories, so definition 2.9 provides the triangle

$$\mathcal{E}_L \star \mathcal{RE} \longrightarrow \mathcal{RE} \xrightarrow{l_{\mathcal{E}}^{\#}} \mathcal{LE} \star \mathcal{RE} \longrightarrow \mathcal{E}_L \star \mathcal{RE}\ [1]. \tag{2.39}$$

We also consider the distinguished triangle of proposition (2.16)

$$\mathcal{LE} \longrightarrow \mathcal{LE} \star \mathcal{RE} \xrightarrow{g_{\mathcal{E}}^{\#}} \mathcal{RE} \longrightarrow \mathcal{LE}\ [1]. \tag{2.40}$$

We start with a lemma.

Lemma 2.18. *The morphism $g^{\#}_{\mathcal{E}} \circ l^{\#}_{\mathcal{E}} \in \mathrm{Hom}_{X\times X}(\mathcal{RE}, \mathcal{RE})$ is an isomorphism.*

Proof. (of the lemma) Since $\mathcal{E}$ is EZ–spherical, proposition 2.8 *iii*) implies that $\mathrm{Hom}_{X\times X}(\mathcal{RE}, \mathcal{RE}) \cong k$, so it is enough to show that the morphism $g^{\#}_{\mathcal{E}} \circ l^{\#}_{\mathcal{E}}$ is non–zero. In fact, we will show that their composition

$$\mathcal{RE} \xrightarrow{l^{\#}_{\mathcal{E}}} \mathcal{LE} \star \mathcal{RE} \xrightarrow{g^{\#}_{\mathcal{E}}} \mathcal{RE},$$

induces group homomorphisms of Hom groups (all isomorphic to $\bar{k}$) by proposition 2.8 $i), iii)$)

$$\mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{g^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{l^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X),$$

that are isomorphisms, where, proposition 2.8 *i*) shows that

$$\mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \cong \bar{k},$$

and, propositions 2.4 and 2.8 *iii*), imply that

$$\mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \cong \mathrm{Hom}_{X\times X}(\mathcal{LE}, \mathcal{LE}) \cong \bar{k}.$$

In order to show that the composition $l^{\#}_{\mathcal{E}} \circ g^{\#}_{\mathcal{E}}$ is non-zero, we will "probe" it with the help of the cohomological functor $\mathrm{Hom}_{X\times X}(\ ?, \Delta_*\mathcal{O}_X)$. First, we apply this functor to the distinguished triangle (2.40), and we look at a piece of the resulting long exact sequence of groups.

$$\begin{aligned} &\ldots \to \mathrm{Hom}_{X\times X}(\mathcal{LE}[1], \Delta_*\mathcal{O}_X) \to \\ &\to \mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{g^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \to \ldots. \end{aligned}$$

But by proposition 2.8 *ii*), the leftmost group is zero, hence the morphism $g^{\%}_{\mathcal{E}}$ is in fact an isomorphism

$$k \cong \mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{\cong} \mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \cong \bar{k}.$$

We now apply the same cohomological functor to the distinguished triangle (2.39) and investigate a piece of the corresponding long exact sequence of groups.

$$\begin{aligned} &\ldots \to \mathrm{Hom}_{X\times X}(\mathcal{E}_L \star \mathcal{RE}[1], \Delta_*\mathcal{O}_X) \to \\ &\to \mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{l^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \to \ldots. \end{aligned}$$

Proposition 2.4 implies that

$$\mathrm{Hom}_{X\times X}(\mathcal{E}_L \star \mathcal{RE}[1], \Delta_*\mathcal{O}_X) \cong \mathrm{Hom}_{X\times X}(\mathcal{E}_L, \mathcal{LE}[-1]) \tag{2.41}$$

and proposition 2.10 shows the the group is zero. Hence the morphism $l^{\%}_{\mathcal{E}}$ is an isomorphism

$$k \cong \mathrm{Hom}_{X\times X}(\mathcal{LE} \star \mathcal{RE}, \Delta_*\mathcal{O}_X) \xrightarrow{\cong} \mathrm{Hom}_{X\times X}(\mathcal{RE}, \Delta_*\mathcal{O}_X) \cong \bar{k},$$

which finishes the proof of the lemma. □

The lemma shows that we can consider the following “9–diagram” (page 24 in [6]) which is essentially a version of the octahedron axiom in triangulated categories.

$$
\begin{tikzcd}
\mathcal{R}\mathcal{E}[1] \arrow[r, "\cong"] & \mathcal{R}\mathcal{E}[1] \arrow[r] & 0 \arrow[r] & \mathcal{R}\mathcal{E}[2] \\
\mathcal{E}_L \star \mathcal{R}\mathcal{E}[1] \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \widetilde{\mathcal{U}} \arrow[r] \arrow[u] & \mathcal{E}_L \star \mathcal{R}\mathcal{E}[2] \arrow[u] \\
\mathcal{L}\mathcal{E} \star \mathcal{R}\mathcal{E} \arrow[r, "g^{\#}_{\mathcal{E}}"] \arrow[u] & \mathcal{R}\mathcal{E} \arrow[r] \arrow[u] & \mathcal{L}\mathcal{E}[1] \arrow[r] \arrow[u] & \mathcal{L}\mathcal{E} \star \mathcal{R}\mathcal{E}[1] \arrow[u] \\
\mathcal{R}\mathcal{E} \arrow[r, "\cong"] \arrow[u, "l^{\#}_{\mathcal{E}}"] & \mathcal{R}\mathcal{E} \arrow[r] \arrow[u, "Id"] & 0 \arrow[r] \arrow[u] & \mathcal{R}\mathcal{E}[1] \arrow[u]
\end{tikzcd}
\tag{2.42}
$$

The starting point is the commutative square located in the lower left corner. The “9–diagram” proposition (prop. 1.1.11. in [6]) shows that we can fill in the diagram, so $\mathcal{E}_L \star \mathcal{R}\mathcal{E}[2] \cong \widetilde{\mathcal{U}} \cong \mathcal{L}\mathcal{E}[1]$. This ends the proof of the first isomorphism of the proposition.

The proof of the second one is very similar. Of course, we have to replace the distinguished triangles (2.39) and (2.40) by

$$
\mathcal{R}\mathcal{E} \star \mathcal{E}_L \longrightarrow \mathcal{R}\mathcal{E} \xrightarrow{l'^{\#}_{\mathcal{E}}} \mathcal{R}\mathcal{E} \star \mathcal{L}\mathcal{E} \longrightarrow \mathcal{R}\mathcal{E} \star \mathcal{E}_L\ [1],
$$

$$
\mathcal{L}\mathcal{E} \longrightarrow \mathcal{R}\mathcal{E} \star \mathcal{L}\mathcal{E} \xrightarrow{g'^{\#}_{\mathcal{E}}} \mathcal{R}\mathcal{E} \longrightarrow \mathcal{L}\mathcal{E}\ [1],
$$

and then show that the morphism $g'^{\#}_{\mathcal{E}} \circ l'^{\#}_{\mathcal{E}} \in \mathrm{Hom}_{X \times X}(\mathcal{R}\mathcal{E}, \mathcal{R}\mathcal{E})$ is an isomorphism.

Everything works as above– the only modification is that the isomorphism (2.41) has to be replaced by

$$
\mathrm{Hom}_{X \times X}(\mathcal{R}\mathcal{E} \star \mathcal{E}_L[1], \Delta_* \mathcal{O}_X) \cong \mathrm{Hom}_{X \times X}(\mathcal{E}_L, \mathcal{L}\mathcal{E}[-1]),
$$

which is true due to proposition 2.4.

We are now in the position to finish the proof of the theorem. The argument is similar to the one used to prove the previous proposition. Consider the distinguished triangle (2.30) defining $\mathcal{E}_L$

$$
\mathcal{E}_L \to \Delta_* \mathcal{O}_X \xrightarrow{l_{\mathcal{E}}} \mathcal{L}\mathcal{E} \xrightarrow{u_{\mathcal{E}}} \mathcal{E}_L[1], \tag{2.43}
$$

with a choice of a (non–canonical) morphism $u_{\mathcal{E}}$. Since $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical, we apply the cohomological functor $\mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E},\ ?)$ to this distinguished triangle and write a relevant piece of the associated long exact sequence:

$$
\begin{aligned}
\ldots \to \mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \Delta_* \mathcal{O}_X) &\xrightarrow{l^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \mathcal{L}\mathcal{E}) \xrightarrow{u^{\%}_{\mathcal{E}}} \mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \mathcal{E}_L[1]) \to \\
\to \mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \Delta_* \mathcal{O}_X[1]) &\to \ldots .
\end{aligned}
$$

By proposition 2.8 $ii), iii)$, we know that

$$
\mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \Delta_* \mathcal{O}_X) \cong 0,\ \mathrm{Hom}_{X \times X}(\mathcal{L}\mathcal{E}, \mathcal{L}\mathcal{E}) \cong \bar{k},
$$

and by propositions 2.4 i), 2.17 and 2.8 i) (in this order), we have that

$$\mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L[1]) \cong \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X,\mathcal{R}\mathcal{E}\star\mathcal{E}_L[1]) \cong \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X,\mathcal{L}\mathcal{E}) \cong \bar{k}. \tag{2.44}$$

We conclude that the morphism $u_{\mathcal{E}} \in \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\ [1])$ is a generator of this group.

Let's now look at the distinguished triangle,

$$\mathcal{E}_L \to \mathcal{E}_L\star\mathcal{E}_R \to \mathcal{E}_L\star\mathcal{R}\mathcal{E}[1] \cong \mathcal{L}\mathcal{E} \xrightarrow{v_{\mathcal{E}}} \mathcal{E}_L[1], \tag{2.45}$$

obtained by applying the exact functor $\mathcal{E}_L\star(?)$ to the distinguished triangle (2.30) defining $\mathcal{E}_R$. Due to proposition 2.17, we have that $\mathcal{E}_L\star\mathcal{R}\mathcal{E}[1]\cong\mathcal{L}\mathcal{E}$, so

$$\mathrm{Hom}_{X\times X}(\mathcal{E}_L\star\mathcal{R}\mathcal{E}[1],\mathcal{E}_L\ [1]) \cong \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\ [1]) \cong \bar{k},$$

by (2.44).

We claim that the morphism $v_{\mathcal{E}}$ is a generator of this group (i.e. non–zero). Indeed, after applying the cohomological functor $\mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\ ?)$ to the distinguished triangle (2.45), we can write a relevant piece of the resulting long exact sequence as follows:

$$\ldots \to \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\star\mathcal{E}_R) \to \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\star\mathcal{R}\mathcal{E}[1]) \xrightarrow{v_{\mathcal{E}}^{\%}} \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L[1]) \to \ldots.$$

Proposition 2.17 and proposition 2.8 iii) show that

$$\mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\star\mathcal{R}\mathcal{E}[1]) \cong \mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{L}\mathcal{E})) \cong \bar{k}.$$

Moreover, we can write that

$$\begin{aligned}\mathrm{Hom}_{X\times X}(\mathcal{L}\mathcal{E},\mathcal{E}_L\star\mathcal{E}_R) &\cong \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X,(\mathcal{R}\mathcal{E}\star\mathcal{E}_L[1])\star\mathcal{E}_R[-1]) \cong\\ &\cong \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X,\mathcal{L}\mathcal{E}\star\mathcal{E}_R[-1]) \cong \mathrm{Hom}_{X\times X}(\mathcal{R}\mathcal{E},\mathcal{E}_R[-1]),\end{aligned}$$

where the first and the last isomorphisms follow from proposition 2.4, and the middle one from proposition 2.17. Proposition 2.10 says that the last group is zero, which proves indeed that $v_{\mathcal{E}}$ is non–zero.

In other words, we have shown that the morphisms $u_{\mathcal{E}}$ and $v_{\mathcal{E}}$ coincide up to an isomorphism. We can then look at the diagram

$$\begin{tikzcd}
\mathcal{E}_L \arrow[r] \arrow[d, "\cong"] & \Delta_*\mathcal{O}_X \arrow[r] \arrow[d, dotted] & \mathcal{L}\mathcal{E} \arrow[r, "u_{\mathcal{E}}"] \arrow[d, "\cong"] & \mathcal{E}_L[1] \arrow[d, "\cong"] \\
\mathcal{E}_L \arrow[r] & \mathcal{E}_L\star\mathcal{E}_R \arrow[r] & \mathcal{E}_L\star\mathcal{R}\mathcal{E}[1] \arrow[r, "v_{\mathcal{E}}"] & \mathcal{E}_L[1]
\end{tikzcd} \tag{2.46}$$

We have just argued that the right hand square is commutative. The axiom TR3 of a triangulated category implies the existence of the dotted morphism. Another well known property of triangulated categories (see, for example, corollary 4, page 242 in [22]) shows that it is also an isomorphism, which concludes the proof of theorem 2.11. □

3 Applications

In this section, we sample some geometric geometric situations in which EZ–spherical objects arise. The list has no claims of being exhaustive.

Example 3.1. The case $Z = \mathrm{Spec}(\bar{k})$ brings nothing new, namely an object $\mathcal{E} \in \mathrm{D}(E)$ is EZ–spherical if and only if $i_*\mathcal{E} \in \mathrm{D}(X)$ is spherical, and the functors $\Phi_{\mathcal{E}_R}$ and $\Phi_{\mathcal{E}_L}$ are those studied in [40]. This follows immediately by adjunction for the pair $(i_*, i^!)$ and by recalling that, in this case, the functors $\mathbf{R}^\bullet q_* \mathbf{R}\mathcal{H}om_E$ are nothing else but the $\mathrm{Ext}^\bullet$ groups on E.

Example 3.2. Let E be a smooth proper divisor in a smooth quasi–projective variety X, and $q = Id : E \to Z = E$. Then $\theta \cong \omega_E \overset{\mathbf{L}}{\otimes} \mathbf{L}i^*\omega_X^{-1} \cong \omega_{E/X} \cong \nu$. In this case $Y = E \times_Z E \cong E$, and any invertible sheaf $\mathcal{E}$ is EZ–spherical, since

$$\mathbf{R}q_*\mathbf{R}\mathcal{H}om_E(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_Z$$

The object $\mathcal{E}_L$ in $\mathrm{D}(X \times X)$ is $\Delta_*\mathcal{O}(-E)$, and the associated Fourier–Mukai functor is the usual

$$\Phi_{\mathcal{E}_L}(?) = \mathcal{O}(-E) \overset{\mathbf{L}}{\otimes} (?).$$

Example 3.3. Assume that $E = X$. This is the case of the so–called fiberwise Fourier–Mukai functors [13], [44], [1]. The typical example is the case of a flat projective Calabi–Yau ($\omega_X \cong \mathcal{O}_X$) fibration $q : X \to Z$ with a generic Calabi–Yau fiber F of dimension k. Consider an object $\mathcal{E} \in \mathrm{D}(X)$ flat over the base Z.

Assume that the restriction $\mathcal{E}_z$ is a spherical object (in the sense of [40]) for any generic fiber F_z, that the dimensions of $\mathrm{Ext}^i(\mathcal{E}_z, \mathcal{E}_z)$ $(z \in Z)$ are constant functions on Z for all i, and that $q_*\mathbf{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_Z$. The Grauert–Grothendieck theorem ([25] III.12.9) implies that

$$\mathbf{R}^i q_*\mathbf{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}) = 0, \text{ for } 0 < i < k.$$

In this case $\theta = \omega_Z$, and since $q^!\mathcal{O}_Z \cong q^*\omega_Z[k]$, the duality theorem (1.3) (or relative Serre duality) gives that $\mathcal{E}$ is EZ–spherical.

When the generic fiber F has the property that $h^{i,0}(F) = 0$, for $0 < i < k$, any invertible sheaf on X will be an EZ–spherical object. For a flat elliptic fibration with a section $\sigma : Z \to X$, the sheaf $\mathcal{O}_{\sigma(Z)}$ is also EZ–spherical.

The next example shows one possible way of relating our results to the theory of exceptional objects [38].

Example 3.4. Assume that E is a proper smooth divisor in X such that $E \cong F \times Z$ (F comes from either fiber, or Fano), $q : E \to Z$, $\dim F = k$. Assume that θ is an invertible sheaf on Z such that $q^!\theta \cong \omega_{E/X}[k]$ Let $\mathcal{E}'$ be an exceptional object in $\mathrm{D}(F)$, i.e. $\mathrm{Hom}_F(\mathcal{E}', \mathcal{E}') = k$, and $\mathrm{Ext}^c_F(\mathcal{E}', \mathcal{E}') = 0$, for $c > 0$. Then the pull-back of $\mathcal{E}'$ to E by the natural projection is an EZ–spherical object in $\mathrm{D}(E)$.

Example 3.5. Let's also discuss in more detail the example that provided the inspiration for this work. Of course, it can generalized to cover more general situations in birational geometry; however, this example captures the essential features. We study the EZ–spherical objects that arise when one considers a smooth Calabi–Yau complete intersection X in a toric variety $\tilde{X}$ and a toric elementary

contraction $\widetilde{q} : \widetilde{X} \to \widetilde{Y}$ in the sense of Mori theory (as studied by M. Reid in [37]). As mentioned in the introduction, such transformations provide geometric models for the physical phase transitions. Let $\widetilde{E}$ represent the loci where $\widetilde{q}$ is not an isomorphism (the exceptional locus) and $\widetilde{Z} := \widetilde{q}(\widetilde{E})$. Corollary 2.6 in [37] shows that $\widetilde{E}$ is a complete intersection of toric Cartier divisors $D_1, \dots, D_d$, and the restriction of $\widetilde{q}$ to $\widetilde{E}$ is a flat morphism whose fibers are weighted projective spaces.

Let E denote the exceptional locus of the restriction of the contraction to X and $q : E \to Z$ the corresponding morphism. We make the assumption that

$$E = X \cap \widetilde{E},$$

and that q is a flat morphism with the fibers F (regular) projective spaces of dimension k. For more on the geometrical details of the situation, the reader may want to consult section 4.4 in [26].

According to proposition 2.7 in [37] we have that

$$D_c \cdot C_\sigma < 0, \tag{3.1}$$

for all $c, 1 \leq c \leq d$, and for any curve C_σ in the class of the contraction, therefore any rational curve contained in a fiber $F \cong \mathbb{P}^k$.

On the other hand, the adjunction formula implies that

$$\omega_F = \mathcal{O}(D_1 + \ldots + D_d)_{|_F}. \tag{3.2}$$

The only way that (3.1) and (3.2) could hold at same time is if all the line bundles of the type $\mathcal{O}(D_{c_1} + \ldots + D_{c_m})_{|_F}$ on $F \cong \mathbb{P}^k$ with $0 < m < d$ are negative of the form $\mathcal{O}(-l)$ with $0 < l < k+1$.

We see that in this case

$$\nu_{|_E} \cong \big(\mathcal{O}(D_1) \oplus \ldots \oplus \mathcal{O}(D_d)\big)_{|_E},$$

and

$$\mathrm{H}^l(F, \Lambda^c \nu_{|_F}) = 0, \text{ for } 0 < l < k+1, 0 < c < d.$$

Definition 2.1 and the Grauert–Grothendieck theorem ([25] III.12.9) imply that any invertible sheaf $\mathcal{L}$ on E is indeed an EZ–spherical object, since $\mathbf{R}\mathcal{H}om_E(\mathcal{L}, \mathcal{L}) = \mathcal{O}_E$ and $\mathbf{R}q_*\mathcal{O}_E = \mathcal{O}_Z$.

Note that there is nothing special about the Calabi–Yau condition. To put the EZ–machinery to work, we only need to know that the restriction to E of the canonical bundle of X is also the pull–back of an invertible sheaf on Z. It should also be noted that, in the course of analyzing this example, the existence of the contraction of X is not needed. Presumably, there could be examples with E a complete intersection of Cartier divisors $D_1, \dots, D_d$ in X and $q : E \to Z$ a flat fibration with a Fano fiber F. In that case, we would only need to impose as an assumption the analog of (3.1) in order to have that any invertible sheaf on E is EZ–spherical.

Remark 3.6. An EZ–spherical object $\mathcal{E}$ has good "portability" properties. Indeed, definition 2.1 has a local character with respect to the embedding $E \hookrightarrow X$. Therefore, if $\mathcal{E}$ is EZ–spherical with respect to a configuration described by a diagram such as (1.1), it will continue to be EZ–spherical if the variety X is replaced (analytically) by the "local" variety given by the total space of the normal bundle $N_{E/X}$, while $q : E \to Z$ is left unchanged. In fact, $\mathcal{E}$ will remain EZ–spherical if X is replaced by any other smooth variety X' such that $N_{E/X'} \cong N_{E/X}$. The explicit passage from X to X' can be realized by using the formal completion of X (or X') along E ([25] II.9), or by the so–called "deformation to the normal cone" ([20] chap. 5).

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